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Hybrid deterministic-statistical analysis of vibro-acoustic systems with domain couplings on statistical components

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Abstract

Recently a hybrid analysis method has been developed in which the various components of a complex vibro-acoustic system can be modeled either deterministically or statistically. The coupling between these two types of component is effected by using a diffuse field reciprocity relation, which relates the cross-spectrum of the forces at the boundary of a statistical component to the vibrational energy level of the component. In practical applications, components may be coupled over a domain, rather than just at the component boundaries—for example, the coupling between a plate and an acoustic volume involves the surface of the plate. It is shown in the present work that the reciprocity relation and the hybrid method can be extended to this situation, so that, for example, it is possible to couple a statistical model of a plate to a finite element model of an acoustical cavity, or to a statistical acoustical cavity. The coupling of statistical acoustical and structural components is not new, but rather is done routinely within statistical energy analysis (SEA); this part of the present work demonstrates that the relevant coupling loss factors can be found by using the hybrid method. The analysis is illustrated by various numerical examples.

1. Introduction

The problem of predicting the mid to high frequency response of a complex vibro-acoustic system faces two major difficulties, both of which arise from the fact that the wavelength of deformation of the system components can be relatively short. Firstly, many degrees of freedom are required to capture the detailed deformation of the system, and secondly the response of the system can be sensitive to imperfections, so that manufacturing uncertainties can lead to significant variability in the performance of nominally identical items. A prime example of this situation arises in the automotive industry, where finite element models having several million degrees of freedom are often employed to represent a vehicle [1], while at the same time it is well known that the acoustic performance of vehicles from a production line can be very variable [2]. Recently a hybrid deterministic-statistical modeling technique has been developed to analyze this type of problem [3–5], in which

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parts of the system are modeled by using the finite element method, while other components are modeled as statistical subsystems. Rather than employ a large number of finite element degrees of freedom, each statistical subsystem is modeled by a single energy variable, thus leading to a large reduction in the size of the computational model required. Furthermore, the approach leads to an estimate of the mean [3] and variance [5] of the response of an ensemble of random systems without the need for Monte Carlo simulations. The key enabling feature of the method is a diffuse field reciprocity relation [6,7] which allows statistical subsystems to be coupled to finite element degrees of freedom within the same model. The aim of the present paper is to use an extended form of this relation to model a class of problem that has not previously been considered within the framework of the hybrid method, namely the coupling of statistical structural subsystems to statistical or deterministic acoustical components.

The original form of the diffuse field reciprocity relation [6] allows the cross-spectrum of the forces on the boundary of a statistical subsystem to be written in terms of the subsystem vibrational energy. In this way, the edges of a statistical plate can be coupled to finite element beams, or the faces of a statistical acoustic volume can be coupled to finite element plates. The aim of the hybrid method is that any component within a system can be modeled either by using the finite element method (or some other deterministic method) or as a statistical subsystem. There is potentially a difficulty with this notion if, for example, a statistical plate is to be coupled to a finite element acoustic volume; in this case the coupling is not along the boundary of the statistical subsystem, but rather over the domain of the component, and the original reciprocity relation [6] does not cover this situation. Whereas the original derivation of the reciprocity relation was based on considering wave propagation, it has subsequently been shown in Ref. [7] that the relation can also be derived from modal arguments; this approach lifts the restriction on the way in which the statistical component is coupled to other components and thus allows the problem of domain, or "area", junctions to be addressed. This development is exploited in the present paper to show how area junctions can be used in the hybrid method to develop a wide range of deterministic-statistical models.

The notion of coupling a statistical model of a plate to a statistical model of an acoustic volume is not new, but rather forms one of the earliest problems addressed by statistical energy analysis (for example in Refs. [8–10]). Furthermore, a statistical model of a plate, which can be coupled to either random or deterministic acoustic loading, has been given by Bonilha and Fahy [11]. The novelty of the present work rests in addressing this type of problem through the methodology of the hybrid method, thus allowing general flexibility in the way in which a complex built-up system can be modeled as an assembly of deterministic and statistical components.

The diffuse field reciprocity relation is reviewed in Section 2, with particular attention to the applicability of the relation to area junctions. The hybrid equations are summarized briefly in Section 3, and the application of the method to systems with area junctions is then considered in Section 4 by means of formulating the relevant equations for several example systems. The efficient numerical implementation of the method is also discussed in this section. Several numerical examples are then given in Section 5, where a comparison is made with benchmark finite element results.

2. The reciprocity relation for a random component

2.1. The wave approach

The diffuse field reciprocity relation derived by Shorter and Langley in Ref. [6] is best described by way of example, and to this end, a simple built-up system consisting of two plates which are embedded in a beam framework is shown in Fig. 1. The most direct way of analyzing the dynamic response of this system would be to generate a finite element model, in which the degrees of freedom consist of the displacements at a set of nodes distributed throughout the system. Although in principle this process is straight forward, in practice the plates may be thin and have a relatively short wavelength of vibration, so that many degrees of freedom are needed to capture the detailed spatial pattern of the response. Furthermore, the plates may be sensitive to imperfections, so that the response is significantly random over the ensemble of possible realizations of the system. Progress towards addressing both of these issues can be made by considering a single plate in isolation, as shown in Fig. 2. Rather than consider the motion at all points on the plate, attention is restricted to a set of

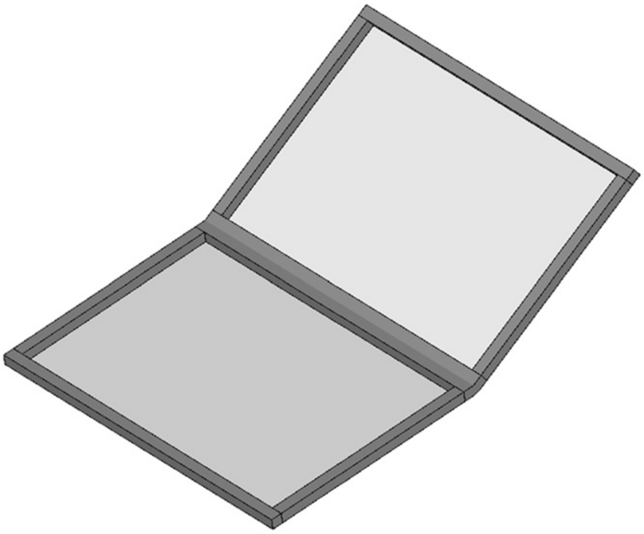

Fig. 1. A built-up system consisting of two plates in a beam framework.

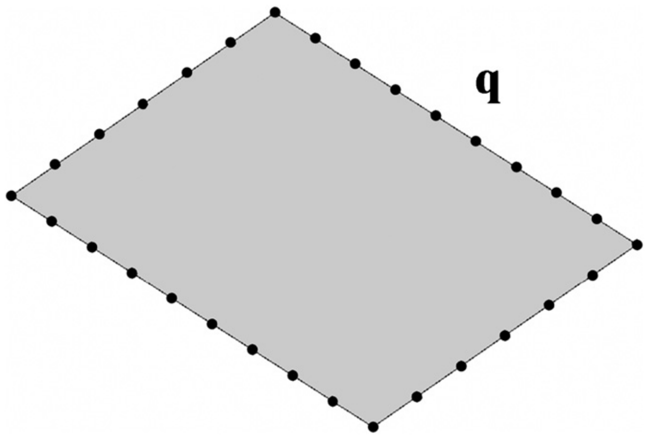

Fig. 2. Boundary degrees of freedom $\mathbf{q}$ for one of the plates in Fig. 1.

degrees of freedom $\mathbf{q}$ which describe the displacement on the boundary. For harmonic motion of frequency ω, the relationship between $\mathbf{q}$ and a corresponding set of external forces $\mathbf{f}$ acting at the boundary can be written in the form

$$\mathbf{Dq} = \mathbf{f}, \tag{1}$$

where $\mathbf{D}$ is the (frequency dependent) dynamic stiffness matrix, and $\mathbf{q}$ and $\mathbf{f}$ are interpreted as complex amplitudes, so that, for example, the time history of the displacement is given by $\mathrm{Re}[\mathbf{q}\exp(\mathrm{i}\omega t)]$. Given $\mathbf{D}$ for each of the two plates in Fig. 1, together with the dynamic properties of the beam framework, it is possible to assemble the dynamic properties of the whole system. The key challenge is to develop a method of computing the properties of $\mathbf{D}$ that avoids the problems associated with the finite element method. In Ref. [6] this is achieved by taking a wave view of the response of the plate: the harmonic motion of the boundary generates elastic waves which propagate across the system and are reflected each time they encounter the boundary. The "direct field" dynamic stiffness matrix $\mathbf{D}_{\mathrm{dir}}$ is defined as the matrix $\mathbf{D}$ which would be obtained were there no reflections, and the difference between the boundary force $\mathbf{D}_{\mathrm{dir}}\mathbf{q}$ produced by this matrix and the actual boundary force $\mathbf{Dq}$ is termed the "reverberant" force, so that $\mathbf{f}_{\mathrm{rev}} = \mathbf{D}_{\mathrm{dir}}\mathbf{q} - \mathbf{Dq}$. With this notation, Eq. (1) can be rewritten in the form

$$\mathbf{D}_{\mathrm{dir}}\mathbf{q} = \mathbf{f} + \mathbf{f}_{\mathrm{rev}}. \tag{2}$$

The matrix $\mathbf{D}_{\mathrm{dir}}$ can be computed efficiently by a variety of methods (see for example [4,6]), so that the aim of providing an efficient description of the plate dynamics will be achieved providing the properties of $\mathbf{f}_{\mathrm{rev}}$ can be found. Based on wave scattering arguments, it was shown in Ref. [6] that if the reverberant wavefield is "diffuse" then

$$\mathrm{E}[\mathbf{f}_{\mathrm{rev}}] = \mathbf{0}, \tag{3}$$

$$\mathrm{E}[\mathbf{f}_{\mathrm{rev}}\mathbf{f}_{\mathrm{rev}}^{*\mathrm{T}}] = \left(\frac{4E}{\omega\pi n}\right)\mathrm{Im}\{\mathbf{D}_{\mathrm{dir}}\}, \tag{4}$$

where E[ ] represents the average taken over an ensemble of random structures, E is the vibrational energy of the plate (defined as twice the time averaged kinetic energy), and n is the plate modal density. Eq. (4) is termed the "diffuse field reciprocity relation", in that it relates the forces $\mathbf{f}_{\mathrm{rev}}$ applied on the boundary by a diffuse wave field to the wave generating properties of the boundary, which are governed by Im $\{\mathbf{D}_{\mathrm{dir}}\}$. The definition of a diffuse wavefield in the present context is fully discussed by Shorter and Langley in [6]; in brief, the ensemble average of the energy stored by each possible wave component must be the same. It was shown in Ref. [6] that Eqs. (2)–(4) enable an efficient method to be developed for the analysis of complex random systems which avoids the use of very large randomized finite element models, as detailed in Section 3.

The derivation of Eq. (4) described above is based on considering wave scattering from the boundaries of a component. However, in some cases a component may be coupled to other components over a domain, rather than just at the boundaries. A common example of this occurs in structural-acoustic coupling, as illustrated in

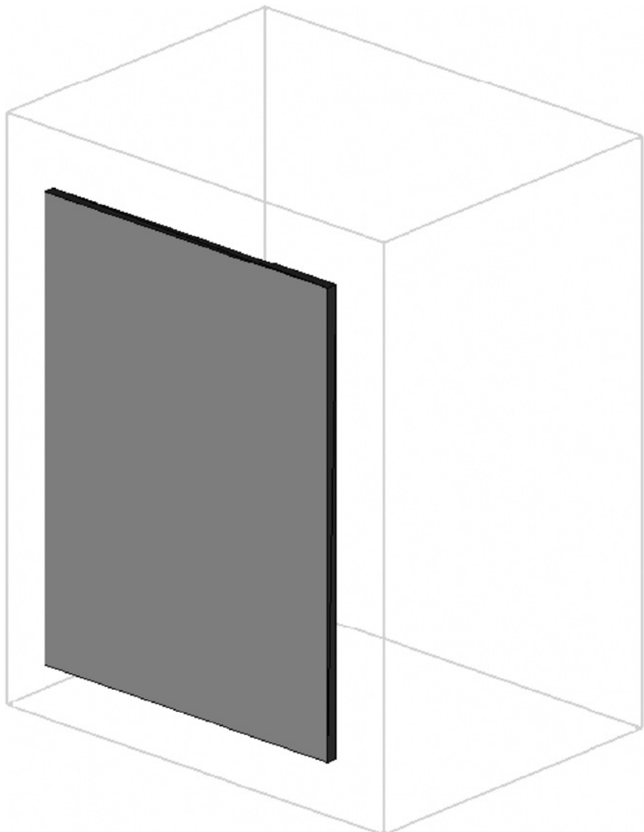

Fig. 3. An example of coupling over a domain: a plate coupled to an acoustic cavity.

Fig. 3. Here a plate is coupled at every point on the surface to an acoustic volume; if the degree of freedom vector $\mathbf{q}$ includes the motion of the interior of the plate, as well as the boundary, then it is not immediately clear how $\mathbf{D}_{\rm dir}$ might be defined, or whether Eq. (4) can be applied to this situation. In this case a modal view of the dynamics of the plate is more instructive than a wave view, as described in the following section.

2.2. The modal approach

Langley in Ref. [7] has presented a derivation of Eq. (4) which is based on modal rather than wave arguments. With this approach, the degrees of freedom $\mathbf{q}$ can form any subset of the total set of degrees of freedom of the component, and in the extreme case may actually coincide with the full set of degrees of freedom. The concept of the direct field dynamic stiffness matrix $\mathbf{D}_{\rm dir}$ is replaced by the ensemble average of the dynamic stiffness matrix, $\mathrm{E}[\mathbf{D}]$, and the reverberant force vector $\mathbf{f}_{\rm rev}$ is replaced by the randomly varying component of the force $\mathbf{f}_{\rm ran}$, so that $\mathbf{f}_{\rm ran} = \mathrm{E}[\mathbf{D}]\mathbf{q} - \mathbf{D}\mathbf{q}$. Eq. (2) is then replaced by

$$\mathrm{E}[\mathbf{D}]\mathbf{q} = \mathbf{f} + \mathbf{f}_{\rm ran}, \tag{5}$$

If the degrees of freedom $\mathbf{q}$ do actually lie on the boundary of the component, then it follows from wave arguments that $\mathrm{E}[\mathbf{D}] = \mathbf{D}_{\rm dir}$. In the more general case, the notation $\mathbf{D}_{\rm dir}$ can be retained as an abbreviation for $\mathrm{E}[\mathbf{D}]$, and in addition it is useful to denote the inverse of $\mathbf{D}_{\rm dir}$ by the receptance matrix $\mathbf{H}_{\rm dir}$. With these definitions, it was shown in Ref. [7] that if the natural frequencies and mode shapes of the component conform to the Gaussian orthogonal ensemble (GOE) then

$$\mathrm{E}[\mathbf{f}_{\rm ran}\mathbf{f}_{\rm ran}^{*\mathrm{T}}] = \left(\frac{4E}{\pi\omega n}\right)\mathrm{Im}\{\mathbf{D}_{\rm dir}\} + \left(\frac{2}{\pi m}\right)[2\,\mathrm{Re}\{\mathbf{S}_{\hat{f}\hat{f}}\} + q(m)\mathbf{S}_{\hat{f}\hat{f}}], \tag{6}$$

$$\mathbf{S}_{\hat{f}\hat{f}} = \mathrm{E}[\hat{\mathbf{f}}\hat{\mathbf{f}}^{*\mathrm{T}}], \tag{7}$$

$$\hat{\mathbf{f}} = \mathrm{i}\mathbf{D}_{\rm dir}\,\mathrm{Im}\{\mathbf{H}_{\rm dir}\}\mathbf{f}, \tag{8}$$

$$q(m) = -1 + \left(\frac{1}{2\pi m}\right)(1 - \mathrm{e}^{-2\pi m}) + \mathrm{E}_1(\pi m)\left[\cosh(\pi m) - \left(\frac{1}{\pi m}\right)\sinh(\pi m)\right], \tag{9}$$

where $m = \omega\eta n$ is the modal overlap of the component, and E_1 is the exponential integral [12]. The GOE is a particular class of random matrix whose eigenvalue and eigenvector statistics have been derived by Mehta [13]. It has been found that these eigen statistics are shared by many types of random system, even though the system matrices do not conform to the GOE, and this phenomenon is referred to as universality. The evidence for the occurrence of GOE eigenvalue and eigenvector statistics in structural dynamics has been discussed by (for example) Mehta [13], Weaver [14], and Langley and Cotoni [15]. In broad terms, it has been found that GOE statistics tend to apply providing the system is random enough for natural frequencies to vary by more than the mean frequency spacing, and this tends to occur in many cases beyond the first few modes. In the vast majority of cases the first term in Eq. (6) is dominant, so that Eq. (6) is in agreement with Eq. (4); this is certainly true when the component carries a diffuse field, in the sense that the ensemble average of the modal energy is the same for each mode. It then follows that Eq. (4) can be applied regardless of whether the degrees of freedom $\mathbf{q}$ lie on the boundary of the component, although some general method of computing the matrix $\mathbf{D}_{\rm dir} = \mathrm{E}[\mathbf{D}]$ is required. Langley [7] has shown that $\mathrm{E}[\mathbf{D}]$ is independent of the damping in the system and furthermore that the variance of $\mathbf{D}$ decreases with increasing damping. Thus for large damping, the matrix $\mathbf{D}$ becomes deterministic and approaches $\mathrm{E}[\mathbf{D}]$. It thus follows that $\mathrm{E}[\mathbf{D}]$ can readily be computed by considering simple physical analogies based on a highly damped system. When $\mathbf{q}$ lies on the boundary of the component, this approach coincides with the wave based estimate $\mathbf{D}_{\rm dir}$, since waves generated at the boundary (the direct field) are damped out prior to the first reflection. The case in which $\mathbf{q}$ lies in the interior of the component is discussed in the following section.

2.3. Area junctions

For the situation shown in Fig. 3, the boundary condition between the plate and the acoustic volume involves the out-of-plane displacement of the whole plate; this type of coupling is referred to here as an ‘‘area junction’’. In this case $\mathbf{q}$ can be defined (for example), as the out-of-plane response at a grid of points covering the surface of the plate. In order to employ Eq. (4) for the plate, it is first necessary to identify the matrix $\mathbf{D}_{\mathrm{dir}}$. Following the argument that a simple physical system corresponding to a non-reverberant or highly damped plate should be selected, an obvious choice of system is an infinite plate, since this eliminates any reflections from the plate boundary. In this case the jkth entry of the receptance matrix is given by

$$H_{\mathrm{dir},jk} = G(r_{jk}), \tag{10}$$

where r_{jk} is the distance between the grid points j and k, and G is the Greens function of the infinite plate. The dynamic stiffness matrix $\mathbf{D}_{\mathrm{dir}}$ is then given by the inverse of $\mathbf{H}_{\mathrm{dir}}$. It is interesting to consider the implications of Eqs. (5), (6) and (10) regarding the behavior of the plate when it is subject to direct excitation $\mathbf{f}$, in which case Eq. (5) yields

$$\mathbf{q} = \mathbf{H}_{\mathrm{dir}}\mathbf{f} + \mathbf{H}_{\mathrm{dir}}\mathbf{f}_{\mathrm{ran}}. \tag{11}$$

The first term on the right hand side of this equation represents the response of an infinite plate to the applied loading, while the second term represents the additional response due to the finite size of the system. For a lightly damped system the second term can be expected to dominate, and Eq. (11) in conjunction with Eq. (6) then yields

$$\mathbf{S}_{qq} = \mathrm{E}[\mathbf{q}\mathbf{q}^{*\mathrm{T}}] = \mathbf{H}_{\mathrm{dir}}\mathrm{E}[\mathbf{f}_{\mathrm{ran}}\mathbf{f}_{\mathrm{ran}}^{*\mathrm{T}}]\mathbf{H}_{\mathrm{dir}}^{*\mathrm{T}} = \left(\frac{4E}{\pi\omega n}\right)\mathbf{H}_{\mathrm{dir}}\,\mathrm{Im}\{\mathbf{D}_{\mathrm{dir}}\}\mathbf{H}_{\mathrm{dir}}^{*\mathrm{T}}, \tag{12}$$

where Eq. (6) has been employed, with the assumption that the second term in the equation is negligible. Now the following matrix identity holds for any matrix $\mathbf{A}$

$$\mathbf{A}\,\mathrm{Im}\{\mathbf{A}^{-1}\}\mathbf{A}^{*} = -\mathrm{Im}\{\mathbf{A}\}, \tag{13}$$

so that Eq. (12) can be written in the form

$$\mathbf{S}_{qq} = -\left(\frac{4E}{\pi\omega n}\right)\mathrm{Im}\{\mathbf{H}_{\mathrm{dir}}\} \Rightarrow \mathrm{E}[q_j q_k^{*}] = -\left(\frac{4E}{\pi\omega n}\right)\mathrm{Im}\{G(r_{jk})\}. \tag{14}$$

Eq. (14) states that the correlation between the response at node j and that at node k is proportional to the imaginary part of the Greens function between these two points. This is a known result for a diffuse wavefield [16], and the emergence of the result from the present analysis supports the arguments leading to Eq. (10).

3. The hybrid FE–SEA equations

The diffuse field reciprocity relation given by Eq. (4) forms the main building block of the hybrid analysis method presented by Shorter and Langley in Ref. [3]. In this approach a complex system is divided into deterministic components, such as the beams shown in Fig. 1, and statistical components, such as the plates shown in this figure. The deterministic components are modeled by using the finite element method (or some other deterministic technique) with degrees of freedom $\mathbf{q}$, whereas the statistical components (or ‘‘subsystems’’) are each assumed to carry a diffuse wavefield, the intensity of which is characterized by the subsystem vibrational energy. The coupling between the deterministic and statistical components is effected by using Eq. (4), which leads to the following result for the response of the deterministic system

$$\mathbf{S}_{qq} = \mathbf{D}_{\mathrm{tot}}^{-1}\left[\mathbf{S}_{ff} + \sum_k \left(\frac{4E_k}{\omega\pi n_k}\right)\mathrm{Im}\{\mathbf{D}_{\mathrm{dir}}^{(k)}\}\right]\mathbf{D}_{\mathrm{tot}}^{-1*\mathrm{T}}, \tag{15}$$

$$\mathbf{D}_{\mathrm{tot}} = \mathbf{D}_{\mathrm{d}} + \sum_k \mathbf{D}_{\mathrm{dir}}^{(k)}. \tag{16}$$

Here $\mathbf{D}_{\mathrm{d}}$ is the dynamic stiffness matrix of the finite element model, $\mathbf{S}_{ff}$ is the cross-spectrum of the forces applied to the deterministic system, $\mathbf{D}_{\mathrm{dir}}^{(k)}$ is the direct field (or, equivalently, the ensemble average) dynamic stiffness matrix for subsystem k, and E_k and n_k are the vibrational energy and modal density of this subsystem. Everything on the right hand side of Eq. (15) can be computed from the physical properties of the system, apart from the subsystem energies E_k, which must be found by considering a power balance condition for each subsystem. This leads to the following set of additional equations

$$\omega(\eta_j + \eta_{\mathrm{d},j})E_j + \sum_k \omega\eta_{jk}n_j(E_j/n_j - E_k/n_k) = P_j + P_{\mathrm{in},j}^{\mathrm{ext}}, \quad j = 1,2,3\ldots, \tag{17}$$

where

$$\omega\eta_{\mathrm{d},j} = \left(\frac{2}{\pi n_j}\right)\sum_{r,s} \mathrm{Im}\{D_{\mathrm{d},rs}\}(\mathbf{D}_{\mathrm{tot}}^{-1}\,\mathrm{Im}\{\mathbf{D}_{\mathrm{dir}}^{(j)}\}\mathbf{D}_{\mathrm{tot}}^{-1*\mathrm{T}})_{rs}, \tag{18}$$

$$\omega\eta_{jk}n_j = (2/\pi)\sum_{r,s} \mathrm{Im}\{D_{\mathrm{dir},rs}^{(j)}\}(\mathbf{D}_{\mathrm{tot}}^{-1}\,\mathrm{Im}\{\mathbf{D}_{\mathrm{dir}}^{(k)}\}\mathbf{D}_{\mathrm{tot}}^{-1*\mathrm{T}})_{rs}, \tag{19}$$

$$P_{\mathrm{in},j}^{\mathrm{ext}} = (\omega/2)\sum_{r,s} \mathrm{Im}\{D_{\mathrm{dir},rs}^{(j)}\}(\mathbf{D}_{\mathrm{tot}}^{-1}\mathbf{S}_{ff}\mathbf{D}_{\mathrm{tot}}^{-1*\mathrm{T}})_{rs}. \tag{20}$$

All of the symbols which appear in Eqs. (17)–(20) have been previously defined, apart from η_j, the loss factor of subsystem j, and P_j, which represents the power input to subsystem j arising from forces applied directly to the subsystem (in contrast to $P_{\mathrm{in},j}^{\mathrm{ext}}$, which arises from forces applied to the deterministic system). Eq. (17) represents a set of equations in the form of the well known statistical energy analysis (SEA) equations (see for example [8]), with the coefficients η_{jk} being known as coupling loss factors. The complete response of the system is found by solving Eq. (17) to yield the subsystem energies, following which Eq. (15) is used to yield the response of the deterministic system.

Full details of the derivation of Eqs. (15)–(20) have been given by Shorter and Langley in Ref. [3], and the application of these equations to a range of examples has been described by Cotoni et al. in Ref. [4]. The results yielded by the equations represent ensemble averaged quantities, and a further set of equations have been derived to predict the ensemble variance of the response [5]. For brevity, these further equations are not considered here, although it can be noted that they require no additional information beyond the system matrices which appear in Eqs. (15)–(20).

It can be noted that Eqs. (15) and (17) relate to the response of the system at a particular frequency ω, and all of the averaging involved relates to ensemble averaging rather than averaging across a frequency band. The equations therefore apply directly to harmonic loading. For random loading (i.e. loads that vary randomly in time), the equations yield the power spectra of the response at frequency ω, and frequency band integration of these spectra will yield mean squared values. In this context it can be noted that SEA is sometimes considered to be a theory governing frequency band averaged quantities; in fact, the fundamental principle is ensemble averaging [17], which equates to frequency band averaging if the response is locally ergodic.

Eqs. (15)–(20) have not previously been applied to systems with area junctions on statistical structural subsystems, and this type of problem is considered in the following section.

4. Application of the hybrid method to systems with area junctions

4.1. General comments

The aim of this section is to demonstrate how the hybrid method may be used to analyse systems in which there are area junctions, i.e. systems in which two-dimensional statistical components are connected to other components (either statistical or deterministic) over a surface domain, rather than at a line boundary. A number of illustrative examples are selected to highlight key aspects of the modeling process, so that it should then be clear how the method can be applied to any other problem. Further examples are then implemented numerically in Section 5, where the hybrid method is benchmarked against a full finite element solution.

4.2. Example 1: a statistical model of a transmission suite

The first example concerns two acoustic volumes which are separated by a flat plate, as illustrated schematically in Fig. 4. The aim is to build a hybrid model in which both volumes, and the plate, are modeled as statistical subsystems, i.e. to produce a statistical energy analysis (SEA) model of the system. To develop a hybrid model, it is necessary to identify a deterministic system which serves to couple the various statistical subsystems, and in this example a ''virtual'' deterministic system is introduced, consisting of a grid of points covering the surface of the plate, with out-of-plane displacements $\mathbf{q}$. The ''finite element model'' associated with these degrees of freedom is null, so that $\mathbf{D}_\mathrm{d} = \mathbf{0}$ in the hybrid equations. In order to build the hybrid model it is necessary to compute the dynamic stiffness matrix $\mathbf{D}_\mathrm{dir}^{(k)}$ for each of the three subsystems in the model, so that Eqs. (17)–(20) can be applied, and these matrices are described in what follows.

The matrix $\mathbf{D}_\mathrm{dir}^{(1)}$ is the direct field dynamic stiffness matrix of the left hand room, as seen by the grid of points. This can be found by considering the grid of points to be embedded in an infinite planar baffle, and then identifying $\mathbf{D}_\mathrm{dir}^{(1)}$ as the dynamic stiffness matrix associated with an acoustic half-space on one side of the baffle. Various methods are available for computing this dynamic stiffness matrix, including the Fourier transform approach [18] and an approach based on jinc functions [19]. The direct field dynamic stiffness matrix for the right hand room $\mathbf{D}_\mathrm{dir}^{(3)}$ follows from the same methodology. The dynamic stiffness matrix of the plate, $\mathbf{D}_\mathrm{dir}^{(2)}$, has been discussed in Section 2.3, and is given by inverting the receptance matrix which appears in Eq. (10). The Greens function is given by

$$G(r_{jk}) = (-\mathrm{i}/8Dk^2)[\mathrm{H}_0^{(2)}(kr_{jk}) - \mathrm{H}_0^{(2)}(-\mathrm{i}kr_{jk})], \tag{21}$$

where D is the flexural rigidity of the plate, k is the (frequency dependent) wavenumber, and $H_0^{(2)}$ is the Hankel function of the second kind of order zero.

The set of coupling loss factors which are yielded by the hybrid method are shown schematically in Fig. 5, where it should be noted that η_{jk} also implies the presence of the coupling loss factor η_{kj}. In detail, the coupling loss factors are given by Eq. (19) with

$$\mathbf{D}_\mathrm{tot} = \mathbf{D}_\mathrm{dir}^{(1)} + \mathbf{D}_\mathrm{dir}^{(2)} + \mathbf{D}_\mathrm{dir}^{(3)}. \tag{22}$$

To consider the physics embodied in Fig. 5, suppose that subsystem 1, the left hand volume, carries an energy E_1. Considering the energy flow arising from this energy alone, it follows from Eq. (17) that there is an energy

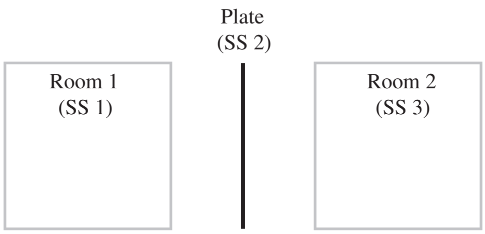

Fig. 4. Schematic of two statistical acoustic volumes separated by a statistical flat plate (SS J represents subsystem J).

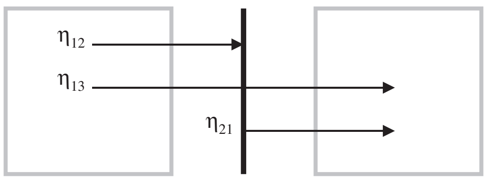

Fig. 5. Schematic of the coupling loss factors yielded by the hybrid method for the system shown in Fig. 4.

flow of $\omega\eta_{31}n_3(E_1/n_1)$ into the right hand volume, the transmission path being via the response of an infinite plate to the applied pressure loading. Below the coincidence frequency, the resulting "non-resonant" motion of the plate will be dominated by mass effects, and thus η_{31} can be expected to encompass "mass law" transmission. Eq. (17) also predicts that there will be an energy flow of $\omega\eta_{21}n_2(E_1/n_1)$ into the plate subsystem, which represents excitation of the plate resonant modes by the acoustic pressure. Thus non-resonant plate effects such as mass law transmission will be included in η_{31}, while the resonant response of the plate is covered by η_{21} and also η_{32}. All of these features appear in the conventional SEA approach to modeling the sound transmission problem [9,10], where η_{21} and η_{32} are calculated from the radiation efficiency of the resonant plate modes, and η_{31} is calculated from the mass law. A key point is that given the reciprocity relation for an area junction, the hybrid method automatically generates these effects.

It can be noted that the hybrid method assumes that each statistical component in the model carries a diffuse wavefield, and so no special attention is given to the boundary conditions acting on the edges of the plate—thus η_{32} does not account for the effect of specific boundary conditions (for example a simply supported edge, or a clamped edge) on the edge radiation below the coincidence frequency. In principle, correction factors could be included to account for a particular boundary condition, or alternatively at low frequencies the plate could be modeled by using the finite element method, and thus form part of the deterministic system rather than a statistical subsystem. These issues are considered numerically in Section 5.

4.3. Example 2: statistical room-plate-cavity-plate-room model

This example represents an extension of the previous example to sound transmission through a double panel, as shown schematically in Fig. 6. The aim is to develop a five subsystem SEA model of the system by using the hybrid method. In this case the "virtual" deterministic system consists of a grid of out-of-plane displacements on the left hand plate, $\mathbf{q}_1$ say, together with a similar set of degrees of freedom on the right hand plate, $\mathbf{q}_2$ say. Again $\mathbf{D}_\mathrm{d} = \mathbf{0}$ for this problem, and in this case the matrix $\mathbf{D}_\mathrm{tot}$ has the following structure when written in terms of the coordinates $(\mathbf{q}_1^\mathrm{T}\mathbf{q}_2^\mathrm{T})^\mathrm{T}$:

$$\begin{aligned}\mathbf{D}_\mathrm{tot} &= \mathbf{D}_\mathrm{dir}^{(1)} + \mathbf{D}_\mathrm{dir}^{(2)} + \mathbf{D}_\mathrm{dir}^{(3)} + \mathbf{D}_\mathrm{dir}^{(4)} + \mathbf{D}_\mathrm{dir}^{(5)} \\ &= \begin{pmatrix} \mathbf{D}_{\mathrm{dir},11}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{D}_{\mathrm{dir},11}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{D}_{\mathrm{dir},11}^{(3)} & \mathbf{D}_{\mathrm{dir},12}^{(3)} \\ \mathbf{D}_{\mathrm{dir},21}^{(3)} & \mathbf{D}_{\mathrm{dir},22}^{(3)} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\mathrm{dir},22}^{(4)} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\mathrm{dir},22}^{(5)} \end{pmatrix}. \end{aligned} \tag{23}$$

The matrices $\mathbf{D}_{\mathrm{dir},11}^{(1)}$ and $\mathbf{D}_{\mathrm{dir},22}^{(5)}$ are given by considering acoustic half-spaces, as in the previous section, while $\mathbf{D}_{\mathrm{dir},11}^{(2)}$ and $\mathbf{D}_{\mathrm{dir},22}^{(4)}$ are the dynamic stiffness matrices associated with infinite plates. The matrix $\mathbf{D}_\mathrm{dir}^{(3)}$ for the cavity can be derived by considering the cavity to be of infinite dimension in the plane of the plates, but of finite thickness. Outside of the domain of the plates, the walls of the cavity can taken to be blocked, and the resulting acoustic dynamic stiffness matrix can then be found by Fourier transform methods. The cavity is an interesting case of the application of the diffuse field reciprocity relation. Langley [7] has shown that for a sufficiently random subsystem this relation is applicable to any subset of the full set of degrees of freedom; imagining the cavity to be modeled by using a set of nodal degrees of freedom on a three dimensional mesh, the subset here consists of those degrees of freedom which are connected to the plates. The matrix $\mathbf{D}_\mathrm{dir}^{(3)}$ is found

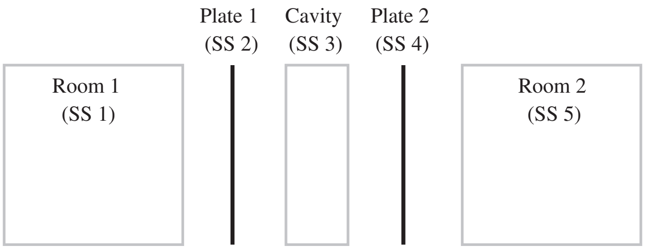

Fig. 6. Schematic of a double wall transmission problem (SS J represents subsystem J).

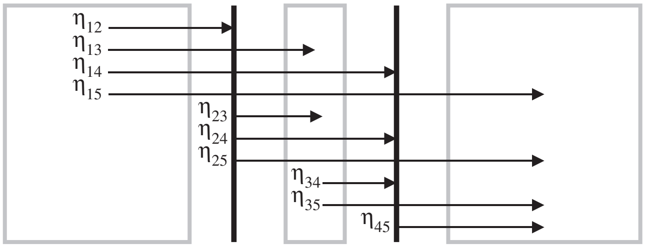

Fig. 7. Schematic of the coupling loss factors yielded by the hybrid method for the system shown in Fig. 6.

by considering a system in which there is no wave reflection from any boundaries other than the plates (since such reflections average to zero in the ensemble averaging process), and this leads to the infinite (in the plane of the plates) cavity described above. This is also equivalent to a highly damped system in the sense that high damping would prevent the reverberance (multiple reflection) of waves propagating in the plane of the plates, but not affect propagation over the narrow cavity width.

The set of coupling loss factors yielded by Eq. (19) for this system is shown schematically in Fig. 7: in terms of panel ''mass law'' effects, any arrow that passes *through* a panel involves mass law behavior for that panel, while any arrow that starts or ends on a panel is related to the resonant response of that panel. The same reasoning applies to the middle cavity, in the sense that arrows that pass through the cavity involve non-resonant (stiffness) behavior of the cavity. The use of the hybrid method to derive the SEA equations of the system automatically identifies the possible coupling paths, without any danger of ''double accounting'', or conversely omitting, any physical effects. The present approach can be compared with the SEA model of Price and Crocker [20]; effectively the terms $\mathbf{D}_{\mathrm{dir},12}^{(3)}$ and $\mathbf{D}_{\mathrm{dir},21}^{(3)}$ are assumed to be zero in that work, so that there are no non-resonant coupling paths through the cavity—such paths would be expected if the cavity is thin compared to the acoustic wavelength. This physical effect has been discussed by Finnveden [21], who has developed an alternative SEA model which incorporates a waveguide model of the double wall system; it was demonstrated that significant errors can arise from neglecting direct transmission paths through the cavity.

As in the previous example, the plate boundary conditions are not considered explicitly in the present approach; correction factors could be included to capture edge radiation effects, or alternatively the plates could be modeled at low frequencies by using the finite element method, rather than modeled as statistical subsystems. The aim of the present section has been to demonstrate how the hybrid methodology allows consistent SEA models to be constructed in an automatic way, and the double wall system serves to highlight aspects of the approach. The numerical examples presented in Section 5 do not cover this system explicitly, but rather consider the reduced case of the single panel system described in the previous section.

4.4. *Example 3: deterministic room-statistical plate-statistical room model*

This example, shown schematically in Fig. 8, is a variation of the first example: the left hand room is modeled by the finite element method rather than as a statistical subsystem, and therefore forms part of the deterministic system. The remaining part of the deterministic system is a ''virtual'' grid of points placed over the plate, with out-of-plane displacements $\mathbf{q}_1$ say. If the degrees of freedom of the finite element model of the room are the nodal pressures $\mathbf{p}$, then the matrix $\mathbf{D}_{\mathrm{d}}$ for this problem has the structure (see for example [22])

$$\mathbf{D}_{\mathrm{d}}\mathbf{q} = \begin{pmatrix} \mathbf{D}_{\mathrm{d},11} & \mathbf{C} \\ \mathbf{C}^{\mathrm{T}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q}_1 \end{pmatrix}. \tag{24}$$

where $\mathbf{D}_{\mathrm{d},11}$ is the dynamic stiffness matrix of the acoustic finite element model, and $\mathbf{C}$ is a coupling matrix, which projects the acoustic pressures onto the plate grid. The matrix $\mathbf{D}_{\mathrm{tot}}$ is then given by

$$\mathbf{D}_{\mathrm{tot}} = \mathbf{D}_{\mathrm{d}} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\mathrm{dir},22}^{(1)} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\mathrm{dir},22}^{(2)} \end{pmatrix}, \tag{25}$$

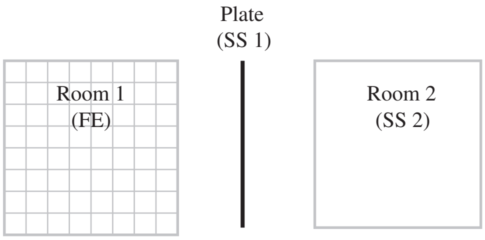

Fig. 8. Schematic of a deterministic acoustic volume separated from a statistical acoustical volume by a statistical flat plate.

where, by analogy with the previous examples, $\mathbf{D}^{(1)}_{\mathrm{dir},22}$ is the dynamic stiffness matrix of an infinite plate, and $\mathbf{D}^{(2)}_{\mathrm{dir},22}$ is the dynamic stiffness matrix of an acoustic half-space. The hybrid method then proceeds as usual, and this results in a coupling loss factor between the plate and the right hand room. The numerical results presented in Section 5 include two examples in which statistical plate components are coupled to finite element models of acoustic cavities.

4.5. Reformulation of the hybrid equations for increased numerical efficiency: "reduced" area junctions

The method described in the previous sections is based on introducing a grid of deterministic degrees of freedom over the surface of each relevant two-dimensional SEA subsystem. The full set of deterministic degrees of freedom in the hybrid model then consists of a grid of freedoms $\mathbf{q}_i$ for each area junction, together with a set of deterministic freedoms, $\mathbf{q}_{\mathrm{m}}$ say, associated with the finite element (FE) model of the deterministic components. The grid freedoms can be grouped so that the various sets $\mathbf{q}_i$ are coupled only through the finite element model. It is possible that more than one SEA subsystem may be attached to a particular set $\mathbf{q}_i$ (for example, an SEA plate and an SEA acoustic volume share a common grid for the case considered in Section 4.4) so that the general structure of the resulting equations for the full set of deterministic degrees of freedom has the form

$$\mathbf{D}_{\mathrm{d}}\mathbf{q} = \begin{pmatrix} \mathbf{D}_{\mathrm{m}} & \mathbf{C}_1 & \mathbf{C}_2 & \cdots \\ \mathbf{C}_1^{*\mathrm{T}} & \sum_{j=1}^{N_1} \mathbf{D}^{(1)}_{\mathrm{dir},j} & \mathbf{0} & \cdots \\ \mathbf{C}_2^{*\mathrm{T}} & \mathbf{0} & \sum_{j=1}^{N_2} \mathbf{D}^{(2)}_{\mathrm{dir},j} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \mathbf{q}_{\mathrm{m}} \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \cdots \end{pmatrix} = \begin{pmatrix} \mathbf{F}_{\mathrm{m}} \\ \mathbf{F}_1 \\ \mathbf{F}_2 \\ \cdots \end{pmatrix}, \tag{26}$$

where: $\mathbf{D}_{\mathrm{m}}$ is the dynamic stiffness matrix of the FE model, including the "direct field" dynamic stiffness matrix of any subsystems that are attached to the model via non-area junctions (e.g. the line stiffness of an SEA plate coupled via an edge, etc); N_i is the number of subsystems attached to the ith grid of points; $\mathbf{D}^{(i)}_{\mathrm{dir},j}$ is the direct field dynamic stiffness matrix of the jth subsystem attached to the ith grid of points; $\mathbf{C}_i$ is the relevant deterministic coupling matrix—it is assumed that no power is dissipated in the coupling, so that the coupling is Hermitian (in practice the coupling matrix will almost always be real); $\mathbf{F}_i$ comprises the external forces applied directly to the freedoms $\mathbf{q}_i$, together with the reverberant forces arising from the attached subsystems, as in Eq. (2). One numerical difficulty associated with the present approach is that $\mathbf{q}_i$ may contain many degrees of freedom, and hence the direct application of the hybrid method via Eqs. (15)–(20) could involve matrices of an unwieldy dimension, such as $\mathbf{D}_{\mathrm{d}}$ in Eq. (26). Because the various two-dimensional grids are coupled only to the FE degrees of freedom $\mathbf{q}_{\mathrm{m}}$, and not to each other, it is possible to fully discuss the practical numerical aspects of the hybrid method by considering the following restricted form of Eq. (26)

$$\begin{pmatrix} \mathbf{D}_{\mathrm{m}} & \mathbf{C}_1 \\ \mathbf{C}_1^{*\mathrm{T}} & \sum_{j=1}^{N_1} \mathbf{D}^{(1)}_{\mathrm{dir},j} \end{pmatrix} \begin{pmatrix} \mathbf{q}_{\mathrm{m}} \\ \mathbf{q}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_{\mathrm{m}} \\ \mathbf{F}_1 \end{pmatrix}. \tag{27}$$

An immediate reduction in the size of the matrices involved in the approach can be made by rearranging Eq. (27) to yield

$$\left[\mathbf{D}_{\mathrm{m}} - \mathbf{C}_1\left(\sum_{j=1}^{N_1}\mathbf{D}_{\mathrm{dir},j}^{(1)}\right)^{-1}\mathbf{C}_1^{*\mathrm{T}}\right]\mathbf{q}_{\mathrm{m}} = \mathbf{F}_{\mathrm{m}} - \mathbf{C}_1\left(\sum_{j=1}^{N_1}\mathbf{D}_{\mathrm{dir},j}^{(1)}\right)^{-1}\mathbf{F}_1, \tag{28}$$

so that the equations are expressed in terms of the FE degrees of freedom alone. It is helpful to introduce the abbreviated notation

$$\mathbf{D}_{\mathrm{m,tot}} = \mathbf{D}_{\mathrm{m}} - \mathbf{C}_1\mathbf{D}_{\mathrm{dir}}^{(1)-1}\mathbf{C}_1^{*\mathrm{T}}, \tag{29}$$

$$\mathbf{D}_{\mathrm{dir}}^{(1)} = \sum_{j=1}^{N_1}\mathbf{D}_{\mathrm{dir},j}^{(1)}, \tag{30}$$

so that Eq. (28) becomes

$$\mathbf{D}_{\mathrm{m,tot}}\mathbf{q}_{\mathrm{m}} = \mathbf{F}_{\mathrm{m}} - \mathbf{C}_1\mathbf{D}_{\mathrm{dir}}^{(1)-1}\mathbf{F}_1. \tag{31}$$

Eq. (31) represents a reduced form of the deterministic system equations, in which the degrees of freedom $\mathbf{q}_1$ do not appear explicitly. It now remains to derive expressions for the SEA coupling loss factors avoiding the explicit use of these freedoms. Progress towards this end can be achieved by considering the energy flows resulting from a reverberant field in subsystem j. From Eq. (31), the deterministic system response to this loading, $\mathbf{F}_{1,j}$ say, is given by

$$\mathbf{q}_{\mathrm{m}} = -\mathbf{D}_{\mathrm{m,tot}}^{-1}\mathbf{C}_1\mathbf{D}_{\mathrm{dir}}^{(1)-1}\mathbf{F}_{1,j}. \tag{32}$$

Now from the reciprocity equation, Eq. (4), it follows that

$$\mathrm{E}[\mathbf{F}_{1,j}\mathbf{F}_{1,j}^{*\mathrm{T}}] = \left(\frac{4E_j}{\omega\pi n_j}\right)\mathrm{Im}\{\mathbf{D}_{\mathrm{dir},j}^{(1)}\}, \tag{33}$$

so that

$$\mathbf{S}_{qq} = \mathrm{E}[\mathbf{q}_{\mathrm{m}}\mathbf{q}_{\mathrm{m}}^{*\mathrm{T}}] = \left(\frac{4E_j}{\omega\pi n_j}\right)\mathbf{D}_{\mathrm{m,tot}}^{-1}\mathbf{C}_1\mathbf{D}_{\mathrm{dir}}^{(1)-1}\,\mathrm{Im}\{\mathbf{D}_{\mathrm{dir},j}^{(1)}\}\mathbf{D}_{\mathrm{dir}}^{(1)-1*\mathrm{T}}\mathbf{C}_1^{*\mathrm{T}}\mathbf{D}_{\mathrm{m,tot}}^{-1*\mathrm{T}}. \tag{34}$$

This result can be rewritten as

$$\mathbf{S}_{qq} = \left(\frac{4E_j}{\omega\pi n_j}\right)\mathbf{D}_{\mathrm{m,tot}}^{-1}\,\mathrm{Im}\{\mathbf{D}_{\mathrm{red},j}^{(1)}\}\mathbf{D}_{\mathrm{m,tot}}^{-1*\mathrm{T}}, \tag{35}$$

where the *reduced* direct field dynamic stiffness matrix $\mathbf{D}_{\mathrm{red},j}^{(1)}$ is defined as

$$\mathbf{D}_{\mathrm{red},j}^{(1)} = -\mathbf{C}_1\mathbf{D}_{\mathrm{dir}}^{(1)-1}\mathbf{D}_{\mathrm{dir},j}^{(1)*}\mathbf{D}_{\mathrm{dir}}^{(1)-1*\mathrm{T}}\mathbf{C}_1^{*\mathrm{T}}. \tag{36}$$

Assuming that the dynamic stiffness matrices are symmetric, it can be noted that

$$\sum_{j=1}^{N_1}\mathbf{D}_{\mathrm{red},j}^{(1)} = -\mathbf{C}_1\mathbf{D}_{\mathrm{dir}}^{(1)-1}\left\{\sum_{j=1}^{N_1}\mathbf{D}_{\mathrm{dir},j}^{(1)*}\right\}\mathbf{D}_{\mathrm{dir}}^{(1)-1*\mathrm{T}}\mathbf{C}_1^{*\mathrm{T}} = -\mathbf{C}_1\mathbf{D}_{\mathrm{dir}}^{(1)-1}\mathbf{C}_1^{*\mathrm{T}}, \tag{37}$$

which is the dynamic stiffness matrix that appears on the right hand side of Eq. (29); the definition expressed by Eq. (36) thus allows this matrix to be represented as a linear sum of contributions from the various subsystems, as in the standard hybrid equation, Eq. (16). Eq. (29) therefore has exactly the form of Eq. (16), with $\mathbf{D}_{\mathrm{red},j}^{(1)}$ playing the role of the direct field dynamic stiffness matrix for subsystem j, expressed in the degrees of freedom $\mathbf{q}_{\mathrm{m}}$. Now the power input to subsystem k arising from the response given by Eq. (35) can be written as

$$P_k = (\omega/2)\mathbf{q}_1^{*\mathrm{T}}\,\mathrm{Im}\{\mathbf{D}_{\mathrm{dir},k}^{(1)}\}\mathbf{q}_1, \tag{38}$$

$$\mathbf{q}_1 = -\mathbf{D}_{\mathrm{dir}}^{(1)-1}\mathbf{C}_1^{*\mathrm{T}}\mathbf{q}_{\mathrm{m}}, \tag{39}$$

(having used Eq. (27) with $\mathbf{F}_1 = 0$) so that

$$P_k = \left(\frac{2E_j}{\pi n_j}\right) \sum_{\mathrm{r,s}} \mathrm{Im}\{D^{(1)}_{\mathrm{red},k,rs}\}(\mathbf{D}^{-1}_{\mathrm{m,tot}}\,\mathrm{Im}\{\mathbf{D}^{(1)}_{\mathrm{red},j}\}\mathbf{D}^{-1*\mathrm{T}}_{\mathrm{m,tot}})_{rs}. \tag{40}$$

Equating this power to $\omega\eta_{kj}n_k(E_j/n_j)$ then yields a coupling loss factor in exactly the same form as the existing hybrid result, Eq. (19), with, as observed previously, $\mathbf{D}^{(1)}_{\mathrm{red},j}$ playing the role of the direct field dynamic stiffness matrix for subsystem j, expressed in the degrees of freedom $\mathbf{q}_\mathrm{m}$.

It follows from the above analysis that the standard hybrid equations can be implemented using the deterministic freedoms $\mathbf{q}_\mathrm{m}$ alone, without explicit reference to the area grid freedoms $\mathbf{q}_i$, providing the direct field dynamic stiffness matrix associated with subsystem j is identified by Eq. (36). This approach can vastly reduce the size of the matrices involved in Eqs. (15)–(20). An identical approach has previously been used by Shorter [23] to reduce the size of the hybrid equations when trim layers are added to deterministic structural components—in this case the degrees of freedom on the "wetted" surface of the trim are removed from the model.

5. Numerical examples

5.1. The transmission loss of a plate

The first numerical example concerns the problem described in Section 4.2, i.e. a transmission suite in which an elastic plate is located in an otherwise rigid wall between two rooms. The plate is taken to be a square steel plate of thickness 5 mm and side length 0.5 m, and each room is taken to have a volume of 10 m^3. The loss factor of the plate is 0.1, while that of each room is 0.01. The critical frequency of the plate is 2370 Hz, and the plate modal density is such that there are approximately 40 modes below this frequency (0.016 modes/Hz). An acoustic source is placed in the left hand room, and the transmission loss of the plate is defined as [24]

$$\begin{aligned}\mathrm{TL} &= 10\log_{10}(\langle p_1^2\rangle/\langle p_3^2\rangle) - 10\log_{10}[A/(S_3\alpha_3)] \\ &= 10\log_{10}(E_1/n_1) - 10\log_{10}(E_3/n_3) - 10\log_{10}[A/(S_3\alpha_3)],\end{aligned} \tag{41}$$

where $\langle p_1^2\rangle$ and $\langle p_3^2\rangle$represent the spatially averaged pressures in the two rooms (labeled as subsystems 1 and 3), A is the area of the panel, and S_3 and α_3 are, respectively, the surface area and the absorption coefficient of the right hand room (for the present model $S_3\alpha_3$ is specified to give a room loss factor of 0.01 at all frequencies). Three approaches have been used to compute the transmission loss: (i) the hybrid method has been used to develop a three subsystem SEA model of the system using the new area junction approach, as described in Section 4.2; (ii) a benchmark model has been developed in which the plate is modeled by using finite elements and the rooms are modeled as statistical subsystems—this approach uses the existing hybrid methodology and the associated software [3,25] to couple the finite element model of the plate to the statistical rooms, and this results in a two subsystem SEA model, Eq. (17), together with the equations governing the plate response, Eq. (15); (iii) a three subsystem SEA model of the system has been developed using standard existing expressions for the coupling loss factors [10,26]. For each of these three models, the resulting SEA equations can be solved to yield the system response, and Eq. (41) can then be used to compute the panel transmission loss.

Results for the transmission loss obtained by using method (i), the present hybrid approach, are compared with results yielded by method (ii), the finite element plate model, in Fig. 9. Three different boundary conditions have been considered for the finite model of the plate, namely simply-supported, clamped, and free. The level of agreement between the present approach and the finite element method is good, with the finite element results for the various plate boundary conditions tending to enclose the present prediction. The results of the present method are compared with the results of method (iii), a standard SEA model, in Fig. 10. In this case the agreement is within 2 dB at low frequencies and closer at higher frequencies. The standard SEA model is based on the methodology outlined by Craik [10] and employs the plate radiation efficiencies and the direct transmission factor (effectively η_{13}) derived by Leppington [26,27]. The irregularities in the standard SEA results around the critical frequency arise from the equations used for the radiation efficiencies, and this is

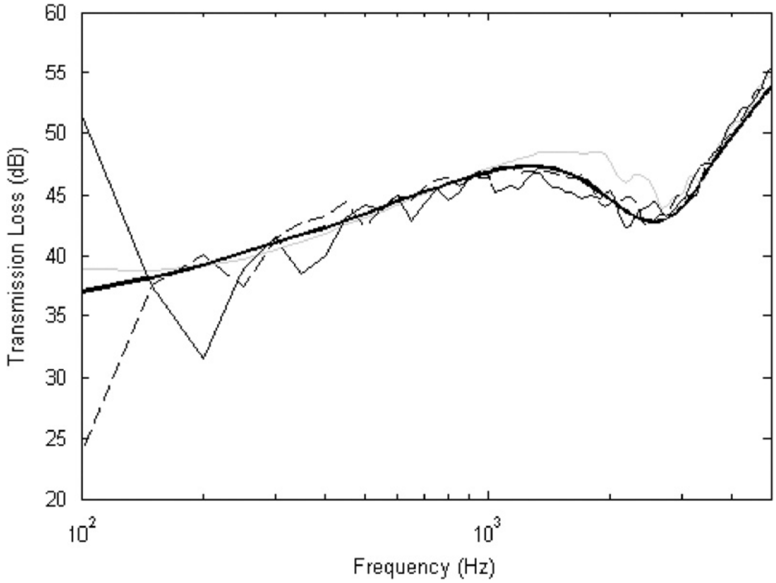

Fig. 9. Transmission loss of a 5 mm thick steel plate from various hybrid models: (——) free-free finite element plate, (- - - - - -) simply-supported finite element plate, (———) clamped finite element plate, and (▬▬▬) statistical plate.

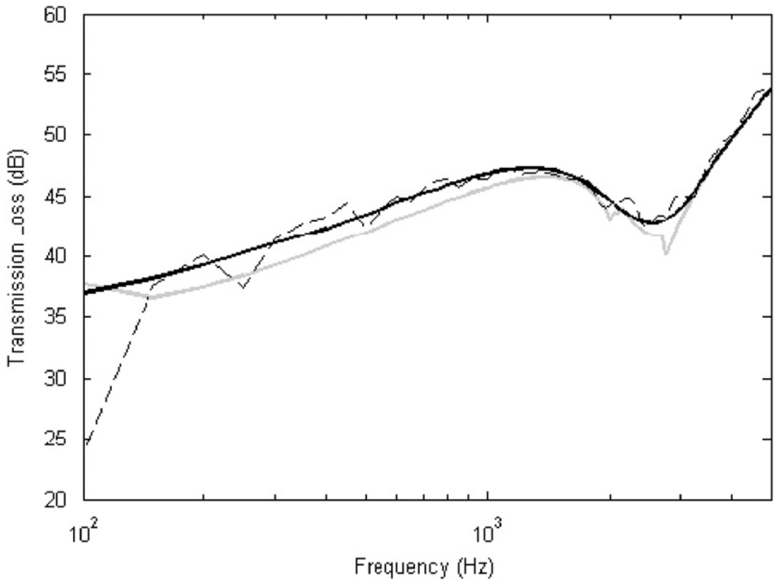

Fig. 10. Transmission loss of a 5 mm thick steel plate: (- - - - - -) hybrid model with a simply-supported finite element plate, (▬▬▬) hybrid model with a statistical plate, and (———) conventional SEA model.

illustrated more clearly in Fig. 11, which shows the coupling loss factors derived by the present method and by the standard SEA approach. The small discontinuities in η_{12} and η_{21} arise from combining the asymptotic formula above and below the critical frequency with a transition formula in the region of the critical frequency [26]. The values of η_{12} and η_{21} yielded by the present approach agree well with those derived from [26], apart from a 2–3 dB discrepancy at low frequencies. This can be traced to the effect of the boundary conditions on acoustic radiation below the critical frequency: the results of [26] employed here relate to simply supported boundary conditions, whereas the present hybrid approach assumes a diffuse wavefield in the plate, without

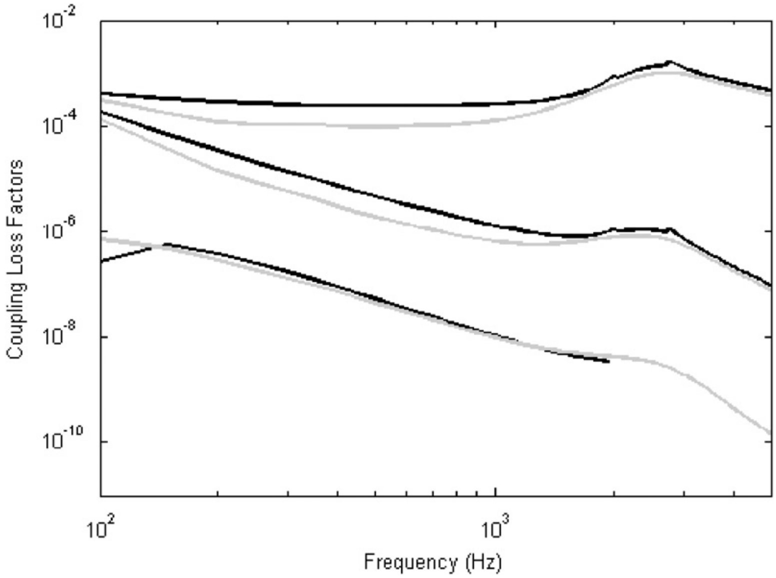

Fig. 11. Coupling loss factors: (———) hybrid formulation, and (▬▬) standard SEA formulation [10,26]. Upper curves η_{21}, middle curves η_{12}, lower curves η_{13}.

special consideration of the plate boundaries. The differences in η_{12} and η_{21} yielded by the two approaches account for the differences in the transmission loss predictions below the critical frequency shown in Fig. 10, which are of the order of 1–2 dB. Also shown in Fig. 10 are the finite element predictions for simply supported edges; it might be excepted that these results should agree more closely with the results of [26] than with the present hybrid theory, but the plate is modally sparse at low frequencies and the results are erratic. The two values of η_{13} shown in Fig. 11 (covering "mass law" effects) show very good agreement below the critical frequency; this effect is not included in the theory of [26] above the critical frequency. It can be concluded from the present example that the hybrid approach captures all of the features of the standard SEA model (aside from the boundary radiation issue), while allowing the relevant coupling loss factors to be computed in a straight forward way.

5.2. A room-plate system

This example forms part of the system considered in Section 4.4: a flexible plate forms one wall of an acoustic cavity, and the cavity is modeled deterministically by using the finite element method. The new area junction approach allows the plate to be modeled as a statistical subsystem, and the results obtained are benchmarked by comparison to a full finite element model of the system generated by the software VA One [25]. The present approach leads to Eq. (15) for the response of the cavity, together with a single SEA equation in the form of Eq. (17) for the response of the plate subsystem. The cavity has dimensions 0.7 m (width) by 1 m (height) by 0.5 m (depth), so that the plate, which is aluminum of thickness 1 mm, has dimensions 0.7 m × 1 m. The loss factors of the plate and the cavity are each 0.01, and the modal density of the plate is 0.223 modes/Hz. The excitation consists of a prescribed velocity source on the rear wall of the cavity (equivalent to a generalized force acting on the finite element pressure degrees of freedom), acting over an area of $8 \times 10^{-4}\,\mathrm{m}^2$ located at the point (0.214,0.252), so that within the present method the excitation of the plate appears through Eq. (20). The finite element mesh employed is shown in Fig. 12, and the results of the present hybrid approach (a statistical plate subsystem combined to a finite element acoustic cavity) are compared with those of a full finite element model in Fig. 13. The full finite element results are shown for five different plate boundary conditions, each of which consists of an irregular combination of clamped and simply supported conditions around the plate perimeter. The present approach predicts the main features of the full results: the peaks in the present results correspond to cavity modes, and there is no appearance of individual plate modes,

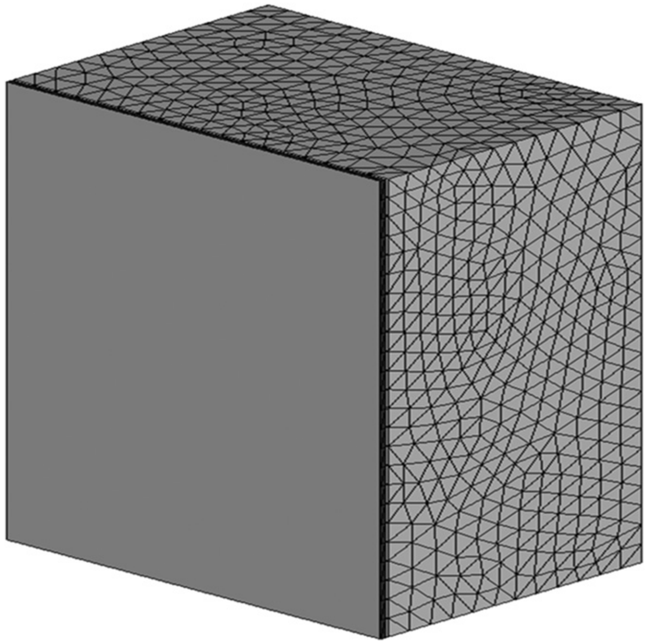

Fig. 12. Statistical plate and finite element cavity considered in the second numerical example.

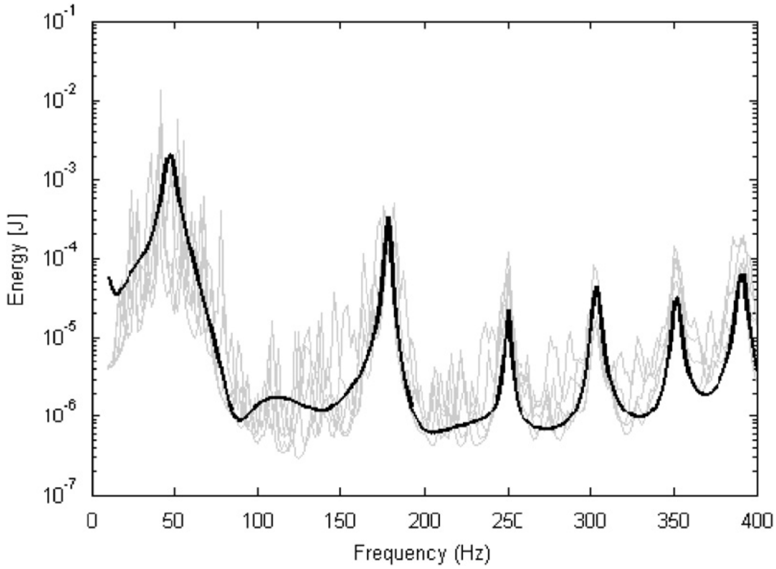

Fig. 13. Energy of the plate for the second numerical example: (———) full finite element model for five different sets of random boundary conditions, and (▬▬▬) hybrid model.

since the effect of these is averaged in the present statistical model. This example demonstrates that the ''area junction'' version of the diffuse field reciprocity relation can be used to couple a statistical structural subsystem to a deterministic acoustic component.

5.3. Vibration transmission through a small acoustic cavity

The final numerical example concerns two plates which are coupled via a small acoustic cavity as shown in Fig. 14(a,b). A force is applied to one of the plates, and the aim is to predict the vibration level in each plate and the acoustic response of the cavity. The plates are each made of aluminum, and have thickness 1 mm and side lengths $0.27\,m \times 0.4\,m$, while the acoustic cavity has thickness 0.1 m (equal to the spacing between the plates), height 0.08 m and width 0.14 m. The modal density of each plate is 0.0689 modes/Hz, and the loss factors for the plates and the cavity are each taken to be 0.01. The system has been modeled by using the new area junction approach to couple two statistical plate subsystems to a finite element model of the cavity (Fig. 14b), and for comparison, benchmark results have been obtained from a finite element model of the whole system (Fig. 14a). The present approach leads to Eq. (15) for the response of the cavity, together with a two subsystem SEA model of the plates, Eq. (17).

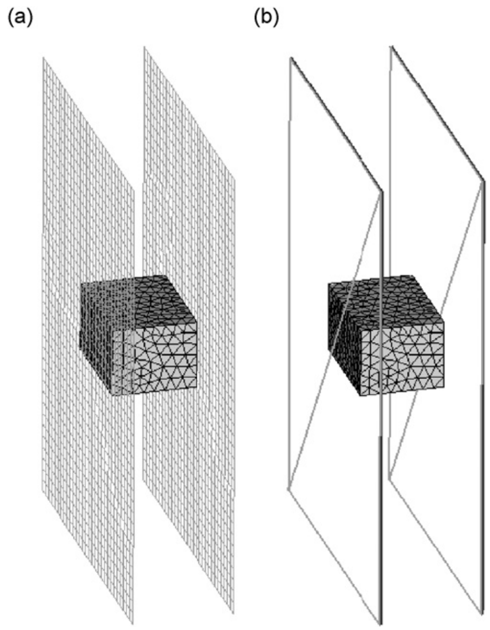

Fig. 14. Two models for analyzing vibration transmission through a small acoustic cavity: (a) left figure, full finite element model, (b) right figure, hybrid model with finite element cavity and statistical plate components.

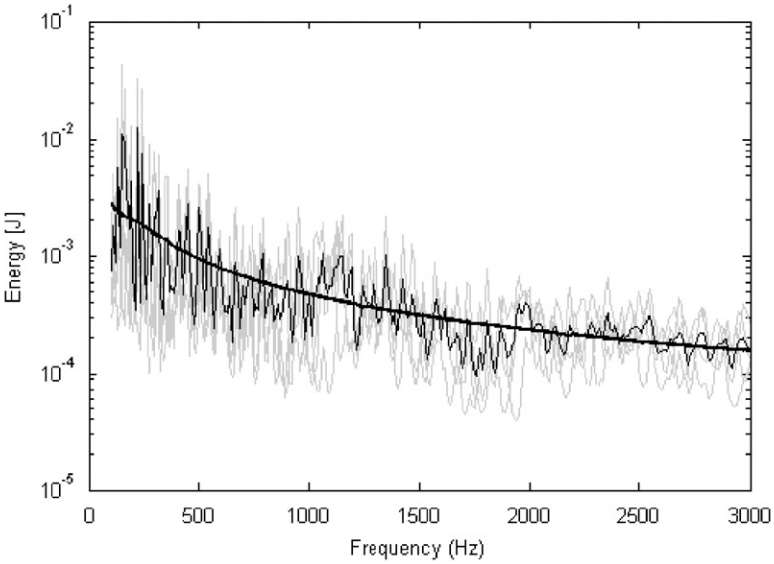

Fig. 15. Energy of the driven plate: (———) full finite element model with four different sets of random boundary conditions, (——) mean response of the four finite element models, and hybrid model (——).

The response of the driven plate is shown in Fig. 15, the response of the cavity is shown in Fig. 16, and the response of the non-driven plate is shown in Fig. 17. Four sets of random plate boundary conditions have been applied to the finite element model of the whole system, and results from each of these systems is shown in the figures, together with the average of the four results. In all cases the present approach yields a good prediction of the system response, with statistical smoothing of the effect of the individual modes in the plates. The use of statistical, rather than deterministic plates, leads to a large reduction in the number of degrees of freedom employed in the model, and a corresponding reduction in the required computer run time.

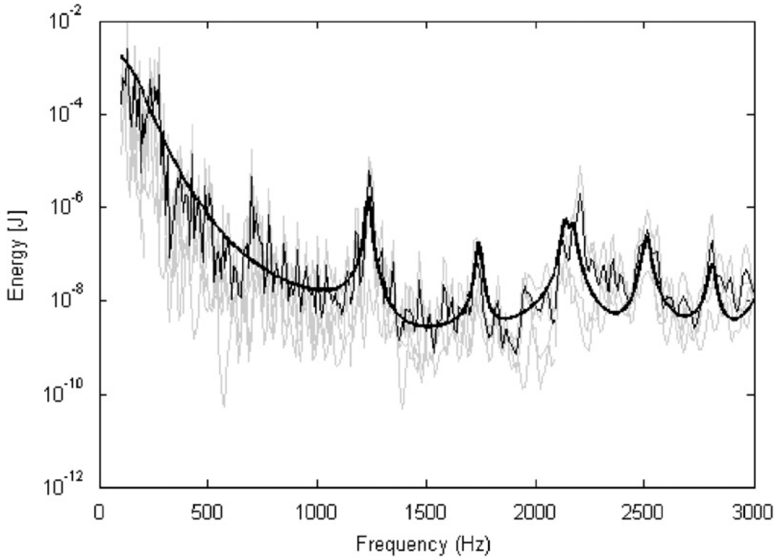

Fig. 16. Energy of the non-driven plate: (———) full finite element model with four different sets of random boundary conditions, (——) mean response of the four finite element models, and hybrid model (▬▬▬).

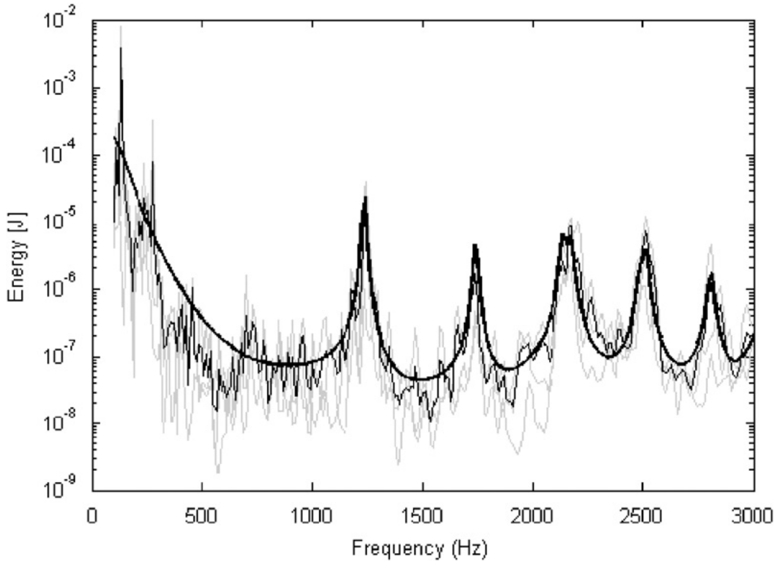

Fig. 17. Energy of the acoustic cavity: (———) full finite element model with four different sets of random boundary conditions, (——) mean response of the four finite element models, and hybrid model (▬▬▬).

6. Conclusions

The key feature of the hybrid method presented by Shorter and Langley [3] is that any component of a complex vibro-acoustic system can be modeled in one of two ways: either (i) deterministically, by using, for example, the finite element method, or (ii) as a statistical subsystem. The coupling between these two disparate types of component is effected by the diffuse field reciprocity relation [6]. Previously this relation has been applied to couplings at the boundaries of the statistical components (for example the edges of a plate), but it has been shown here that the approach can readily be extended to coupling over the domain of a subsystem, termed here an ‘‘area junction’’. This enables, for example, a statistical structural component to be coupled to either a finite element acoustic volume, or (via a grid of ‘‘virtual’’ freedoms) a statistical acoustic volume. Potential industrial applications of this approach include systems such as automotive door panels, where acoustic pockets lie between modally dense structural panels, suggesting the use of a statistical structural model coupled to a deterministic acoustic model. Statistical structural components are routinely coupled to statistical acoustic volumes in SEA and this aspect of the present work is therefore not a new capability; however the present work has shown that this type of modeling is encompassed by the hybrid method.

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